THE CRITICAL NUTRIENT DEFICIENCY LEVEL AND ECONOMIC OPTIMUM NUTRIENT LEVEL FOR THE MITSCHERLICH GROWTH CURVE

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ABSTRACT

Plant growth is modeled with the Mitscherlich response equation. Critical nutrient deficiency levels are defined in various ways including: the nutrient level associated with 100p% of maximum yield for $0<p<1$ (denoted x_p); the nutrient level associated with the point of maximum curvature on the response function (denoted x_c); and, the economic optimum nutrient level under a linear cost function for the nutrient (denoted x_{opt}). The Mitscherlich growth curve is reparameterized to include each of these definitions of a critical nutrient deficiency level. Such a reparameterization enables easy access to approximate standard errors of these critical nutrient levels as well as approximate one- or two-sided confidence intervals. An example is presented using the SAS® procedure for nonlinear least squares.

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INTRODUCTION

Critical nutrient deficiency levels associated with 90% of maximum yield have been determined by both free-hand graphical procedures and the fitting of the Mitscherlich growth equation (Ohki, 1977; Ware, Ohki, and Moon, 1982). Only point estimates have been reported for the nutrient level associated with 90% of maximum yield (denoted x_{90}) with no indication of how to determine a standard error associated with the reported critical nutrient deficiency level. It is shown that a reparameterization of the usual Mitscherlich equation enables determination of an asymptotically valid standard error for the parameter representing the critical nutrient deficiency level. Thus, a useful interval estimate may be calculated for a stated level of confidence.

Current definitions of the critical nutrient deficiency level are arbitrary and subject to criticism. A new definition is proposed. The definition is based upon determining the nutrient level associated with the point of maximum curvature on the Mitscherlich growth curve and is denoted by x_c. This formulation provides an estimate distinct from that provided by the estimate of x_{90}.

Another potentially useful statistic for the Mitscherlich curve is the economic optimum nutrient level. This is determined by maximizing the response subject to a linear cost function for the nutrient. The resulting optimum, x_{opt}, is similar in form to x_c. The Mitscherlich model may again be reparameterized to include x_{opt} as a parameter in the model.

In general, the practice of reparameterizing a nonlinear (or even linear) response equation to include a parameter of research interest may be a profitable one. Not only are asymptotically valid standard errors output by statistical computing software, but estimators are often (though not necessarily) less correlated after reparameterization. This results in faster, more stable convergence to the least squares estimates. Further, the reparameterization results in a model whose parameters are of direct research interest to the investigator and are thus immediately interpretable.

REPARAMETERIZATION

The Mitscherlich growth model is usually parameterized as:

$$Y_i=\beta[1-\gamma\exp(-\alpha x_i)]+\varepsilon_i \ ,$$

where ε_i are independent, homoscedastic, random errors with zero mean. If the errors are normally distributed then nonlinear least squares estimators are also the maximum likelihood estimators (MLEs). The unknown parameters β, α, and γ correspond to the maximum attainable yield of the response as the nutrient level x increases indefinitely, the rate of change in growth, and the fraction of increase attributable to x in the asymptotic yield, respectively.

The critical nutrient deficiency level associated with 90% of maximum yield is easily found to be $x_{90} = [\ln(10\gamma)]/\alpha$. This number is a ratio of functions of the parameters from the Mitscherlich equation. Consequently, determination of a meaningful standard error for the estimator of x_{90} may be a laborious task. Fortunately, some nonlinear regression programs (e.g., BMDP3R and PAR, 1983) will calculate asymptotically valid standard errors for user specified functions of the parameters of a nonlinear regression equation. However, not all commonly used statistical packages with a nonlinear least squares procedure (e.g., SAS PROC NLIN, 1985) will compute the standard errors of estimated functions of the parameters of a nonlinear function. To circumvent the need for tedious hand calculations it is sufficient to reparameterize the growth model as:

$$Y_i=\beta\{1-0.10\exp[-\alpha(x_i-x_{90})]\}+\varepsilon_i \, .$$

This incorporates x_{90} as one of the three parameters in the model to be estimated. The nonlinear least squares estimation procedures of many statistical packages will provide both asymptotic standard errors and asymptotic confidence intervals for the parameter estimates. Thus, a meaningful interval estimate of x_{90} can be reported in addition to the point estimate.

In general, let x_p represent the nutrient level associated with 100p% of the maximum yield. Then for $0<p<1$ the following reparameterization of the Mitscherlich equation incorporates x_p as a model parameter:

$$Y_i=\beta\{1-(1-p)\exp[-\alpha(x_i-x_p)]\}+\varepsilon_i \, .$$

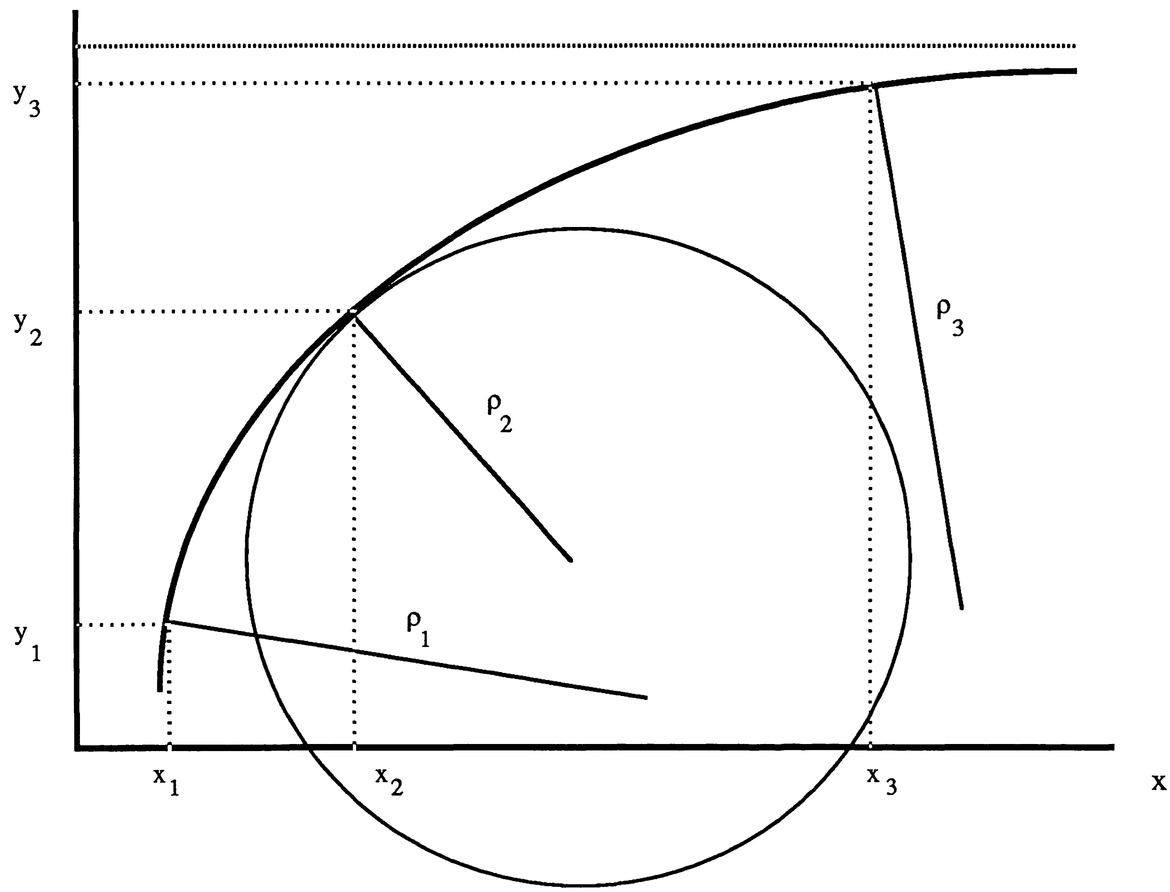
NEW FORMULATION OF CRITICAL LEVEL

Ulrich (1952) considered the critical nutrient concentration for plant growth as: (i) the nutrient concentration which is just deficient or, equivalently, just adequate for maximum growth; or, (ii) the nutrient concentration which separates the deficiency from the adequacy zones.

This definition was criticized because the selection of the desired point just below maximum growth on the curve was arbitrary. Ulrich and Hills (1973) "resolved" this criticism by defining the critical nutrient deficiency level as that nutrient concentration resultant in 90% of maximum growth. This proposed resolution is itself arbitrary. A more precise formulation of the critical nutrient level as defined by Ulrich (1952) is readily obtained if the growth model is known.

Heuristically, the argument proceeds via geometric considerations of the growth curve. The claim is that the critical nutrient deficiency level corresponds to that point on the curve which has the "greatest curvature." Thus, it corresponds to the nutrient level associated with the point on the curve whose osculating circle (an osculating circle is circle tangent to the curve and having the same curvature) has the smallest radius. This is illustrated in the accompanying figure 1. Three radii of osculating circles, ρ_i, are found corresponding to the nutrient levels x_i. The circle corresponding to the second radius is also included in the figure. Notice that ρ_2 is smaller than ρ_1 and ρ_3 indicating that x_2 is closer to the critical nutrient deficiency level than are the x's associated with the other two radii. Thus, supposing x_2 was x_c, the critical nutrient deficiency level, then the radii corresponding to x's both above and below x_c would be longer than ρ_2. Clearly, as x gets indefinitely large the Mitscherlich curve approaches the constant asymptote and the radius of the osculating circle tends to infinity.

Figure 1: The Mitscherlich curve with an osculating circle and several radii.



The above argument may be formalized with some basics of differential geometry (Lipschutz, 1969). The radius of curvature, κ, is defined as the reciprocal of the radius of the osculating circle. Thus, finding the nutrient level associated with minimum radius is equivalent to determining the nutrient level which results in a maximum radius of curvature. The radius of curvature is calculated via the following formula:

$$\kappa=\left|\frac{d^2y}{dx^2}\right|\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-\frac{3}{2}}$$

The usual parameterization of the Mitscherlich curve gives the nutrient level associated with maximum curvature as

$$x_c = [\ln(\alpha\beta\gamma)+(\ln 2)/2]/\alpha\ .$$

A bit of algebra gives: $x_c = x_{90} + [\ln(\alpha\beta)-(\ln 0.02)/2]/\alpha$.

It is easy to conceive of other potential definitions for the critical nutrient deficiency level. However, the formulation presented above appears to be the only one which satisfies the original definition of the critical nutrient deficiency level and has a rigorous derivation.

OPTIMUM NUTRIENT LEVEL VIA COST/BENEFIT RATIO

If one assumes a linear cost function then it is easy to find the nutrient level corresponding to the maximum yield for a minimum cost. Let θ=cost/benefit ratio, the slope of the linear cost function in x. Then the optimum nutrient level, x_{opt}, is given by $x_{opt}=[\ln(\alpha\beta\gamma)-\ln\theta]/\alpha$. This is identical to x_c with (ln2)/2 replaced by $-\ln\theta$ in x_{opt}. Thus, the critical nutrient deficiency level and the optimum nutrient level are the same for $\theta=1/\sqrt{2}$ (about 0.71). Reparameterizing the Mitscherlich equation to include x_{opt} as a model parameter gives:

$$Y_i=\beta-[\theta/\alpha]\exp[-\alpha(x_i-x_{opt})]+\varepsilon_i\ .$$

As before, asymptotically valid standard errors and confidence intervals for x_{opt} will be output from nonlinear regression programs for the model parameterized in this form.

EXAMPLE

The following data are similar to those used in figure 1 of Ware, Ohki, and Moon (1982). The observations represent the response of cotton top growth (grams dry weight) to magnesium (Mn) in tissue (ppm). There are 38 observations used to estimate the parameters and their associated standard errors for the various parameterizations of the Mitscherlich growth curve. The data are listed in the appendix. The SAS® code used to generate the models and the appropriate output are given below.

1. Usual parameterization as: $Y_i = \beta[1 - \gamma \exp(-\alpha x_i)] + \varepsilon_i$

```
PROC NLIN METHOD=GAUSS;
   PARMS B=35 G=.7 A=.25;
   E = EXP(-A*X);
   MODEL Y = B*(1-G*E);
   DER.A    = B*G*X*E;
   DER.B    = 1-G*E;
   DER.G    = -B*E;
```

2. Parameterization with x_p as: $Y_i = \beta\{1 - (1-p)\exp[-\alpha(x_i - x_p)]\} + \varepsilon_i$

```
PROC NLIN METHOD=GAUSS;
   PARMS B=35 A=.25 XP=5;
   P = 0.9;
   E = (1-P)*EXP(-A*(X-XP));
   MODEL Y = B*(1-E);
   DER.B    = 1-E;
   DER.A    = B*E*(X-XP);
   DER.XP = -B*A*E;
```

3. Parameterization with x_{opt} as: $Y_i = \beta - [\theta/\alpha]\exp[-\alpha(x_i - x_{opt})] + \varepsilon_i$

where x_c is a special case of x_{opt} for $\theta = 1/\sqrt{2}$.

```
PROC NLIN METHOD=GAUSS;
   PARMS B=35 A=.25 XOPT=9;
   T = 1/SQRT(2);
   E = EXP(-A*(X-XOPT));
   MODEL Y = B - T*E/A;
   DER.B = 1;
   DER.A = T*E*(X-XOPT+1/A)/A;
   DER.XOPT = -T*E;
```

CONCLUSIONS

It is probably of more interest to compute a one-sided confidence interval for the critical nutrient deficiency level. An approximate one-sided $100\delta\%$ confidence interval for x_{crit} is given by:

$$x_{crit} - z_\delta \, s_{x_{crit}} ,$$

where $\Phi(z_\delta) = \delta$ and $\Phi(.)$ the cumulative distribution function of the standard normal distribution.

APPENDIX

The data used in the example are:

OBS	X	Y	YHAT	RESID
1	1.5	12.0	14.8541	-2.8541
2	2.0	14.4	17.2960	-2.8960
3	2.5	24.2	19.4491	4.7509
4	2.5	25.0	19.4491	5.5509
5	2.5	26.0	19.4491	6.5509
6	2.7	10.5	20.2373	-9.7373
7	2.9	27.2	20.9867	6.2133
8	3.0	12.5	21.3475	-8.8475
9	3.3	23.3	22.3770	0.9230
10	4.0	28.5	24.4972	4.0028
11	4.8	27.0	26.5042	0.4958
12	5.0	30.0	26.9459	3.0541
13	5.5	21.2	27.9576	-6.7576
14	8.0	25.0	31.4801	-6.4801
15	8.0	35.0	31.4801	3.5199
16	8.5	28.5	31.9555	-3.4555
17	10.0	33.5	33.0700	0.4300
18	10.2	36.5	33.1892	3.3108
19	10.5	35.2	33.3573	1.8427
20	12.5	32.5	34.2045	-1.7045
21	14.5	36.8	34.7165	2.0835
22	16.0	36.5	34.9626	1.5374
23	19.0	37.0	35.2469	1.7531
24	20.0	34.5	35.3030	-0.8030
25	57.5	38.0	35.4988	2.5012
26	73.0	35.0	35.4988	-0.4988
27	82.5	37.3	35.4988	1.8012
28	85.0	37.5	35.4988	2.0012
29	87.5	37.5	35.4988	2.0012
30	95.0	36.0	35.4988	0.5012
31	108.0	32.2	35.4988	-3.2988
32	130.0	32.5	35.4988	-2.9988
33	131.0	35.0	35.4988	-0.4988
34	133.0	33.5	35.4988	-1.9988
35	143.0	33.0	35.4988	-2.4988
36	150.0	33.5	35.4988	-1.9988
37	165.0	37.0	35.4988	1.5012
38	180.0	36.5	35.4988	1.0012

LITERATURE CITED

Ohki, K. 1977. Critical zinc levels related to early growth and development of determinate soybeans. Agron. J. 69:969-974.

Ulrich, A. 1952. Physiological bases for assessing the nutritional requirements of plants. Annu. Rev. Plant Physiol. 3:207-228.

———, and F. J. Hills. 1973. Plant analysis as an aid in fertilizing sugar crops: Part 1. Sugar beets. p.271-288. *In* L. M. Walsh and J. D. Beaton (ed.) *Soil testing and plant analysis*. Soil Sci. Soc. of Am., Inc., Madison, Wis.

Ware, G. O., K. Ohki, L. C. Moon. 1982. The Mitscherlich plant growth model for determining critical nutrient deficiency levels. Agron. J. 74:89-91.

BMDP Statistical Software. 1983. *BMDP Statistical Software Manual, 1983 Edition.* Berkeley, CA: University of California Press.

SAS Institute Inc. 1985. *SAS® User's Guide: Statistics, Version 5 Edition.* SAS Institute Inc.

Lipschultz, M.M. 1969. *Theory and Problems of Differential Geometry*. New York, NY: McGraw–Hill.